Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book D

Contents

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2.1

Additional exercises for Unit D1

Section 1

Additional Exercise D1

Arrange the following numbers in increasing order.

(a) $7/36$, $3/20$, $1/6$, $7/45$, $11/60$

(b) $0.\overline{465}$, $0.4\overline{65}$, $0.46\overline{5}$, 0.4655, 0.4656

Additional Exercise D2

Find the fractions whose decimal representations are as follows.

(a) $0.\overline{481}$ (b) $0.48\overline{1}$

Additional Exercise D3

Let $x = 0.\overline{21}$ and $y = 0.\overline{2}$. Use long division to find $x + y$ and xy in decimal form.

Additional Exercise D4

Find a rational number x and an irrational number y, both in the interval $(0.119, 0.12)$.

Section 2

Additional Exercise D5

Solve the following inequalities.

(a) $\dfrac{x-1}{x^2+4} < \dfrac{x+1}{x^2-4}$ (b) $\sqrt{4x-3} > x$

(c) $|17 - 2x^4| \leq 15$

Section 3

Additional Exercise D6

Use the Triangle Inequality to prove that

$$|a| \leq 1 \implies |a-3| \geq 2.$$

Additional Exercise D7

Prove that

$$(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2, \text{ for } a,b,c,d \in \mathbb{R}.$$

Additional Exercise D8

Prove the inequality

$$3^n \geq 2n^2 + 1, \text{ for } n \geq 1,$$

(a) by using the Binomial Theorem, applied to $(1+x)^n$ with $x = 2$

(b) by using mathematical induction.

Additional Exercise D9

Apply Bernoulli's Inequality, first with $x = 2/n$ and then with $x = -2/(3n)$, to prove that

$$1 + \frac{2}{3n-2} \leq 3^{1/n} \leq 1 + \frac{2}{n}, \text{ for } n \geq 1.$$

Additional Exercise D10 Challenging

Use mathematical induction to prove that

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \leq \sqrt{\frac{3/4}{2n+1}}, \text{ for } n \geq 1.$$

Section 4

Additional Exercise D11

In this exercise, take

$$E_1 = \{x \in \mathbb{Q} : 0 \leq x < 1\}$$

and

$$E_2 = \{(1 + 1/n)^2 : n = 1, 2, \ldots\}.$$

(a) Prove that each of the sets E_1 and E_2 is bounded above. Which of them has a maximum element?

(b) Prove that each of the sets E_1 and E_2 is bounded below. Which of them has a minimum element?

(c) Determine the least upper bound of each of the sets E_1 and E_2.

(d) Determine the greatest lower bound of each of the sets E_1 and E_2.

Solutions to additional exercises for Unit D1

Solution to Additional Exercise D1

(a) Putting all the fractions over the common denominator of 180, we obtain

$$\frac{3}{20} < \frac{7}{45} < \frac{1}{6} < \frac{11}{60} < \frac{7}{36}.$$

(Alternatively, we can compare the decimal representations of these fractions.)

(b) $0.\overline{465} < 0.4655 < 0.46\overline{5} < 0.4656 < 0.4\overline{65}$

Solution to Additional Exercise D2

(a) Let $x = 0.\overline{481}$.

Multiplying both sides by 10^3, we obtain

$$1000x = 481.\overline{481} = 481 + x.$$

Hence

$$999x = 431, \text{ so } x = 481/999 = 13/27.$$

(b) Let $y = 0.\overline{1}$.

Multiplying both sides by 10, we obtain

$$10y = 1.\overline{1} = 1 + y.$$

Hence

$$9y = 1, \text{ so } y = 1/9.$$

Thus

$$0.48\overline{1} = \frac{48}{100} + \frac{y}{100} = \frac{48}{100} + \frac{1}{900} = \frac{433}{900}.$$

Solution to Additional Exercise D3

If $x = 0.\overline{21}$, then multiplying both sides by 100 we obtain

$$100x = 21.\overline{21} = 21 + x.$$

Hence

$$99x = 21, \text{ so } x = 21/99.$$

If $y = 0.\overline{2}$, then multiplying both sides by 10 we obtain

$$10y = 2.\overline{2} = 2 + y.$$

Hence

$$9y = 2, \text{ so } y = 2/9.$$

Thus

$$x + y = \frac{21}{99} + \frac{2}{9} = \frac{43}{99} = 0.\overline{43}$$

and

$$xy = \frac{21}{99} \times \frac{2}{9} = \frac{14}{297} = 0.\overline{047\,138}.$$

Solution to Additional Exercise D4

For example,

$$x = 0.1195 \text{ and } y = 0.119\,501\,001\ldots,$$

where $501\,001\ldots$ is a non-recurring tail.

Solution to Additional Exercise D5

(a) Note that this inequality should not be solved by cross-multiplying because $x^2 - 4$ can be positive, zero or negative depending on the value of x. Rearranging this inequality we have

$$\begin{aligned}\frac{x-1}{x^2+4} < \frac{x+1}{x^2-4} &\iff 0 < \frac{x+1}{x^2-4} - \frac{x-1}{x^2+4}\\ &\iff 0 < \frac{2x(x+4)}{(x^2-4)(x^2+4)}\\ &\iff 0 < \frac{2x(x+4)}{(x-2)(x+2)(x^2+4)}.\end{aligned}$$

Using a table of signs, including the factors $x, x+4, x-2, x+2$ and x^2+4, we find that the solution set is

$$\left\{x : \frac{x-1}{x^2+4} < \frac{x+1}{x^2-4}\right\} = (-\infty, -4) \cup (-2, 0) \cup (2, \infty).$$

(b) The expression $\sqrt{4x-3}$ is defined, and non-negative, when $4x - 3 \geq 0$; that is, for $x \geq \frac{3}{4}$.

For $x \geq \frac{3}{4}$, we see that

$$\begin{aligned}\sqrt{4x-3} > x &\iff 4x - 3 > x^2\\ &\iff 0 > x^2 - 4x + 3\\ &\iff 0 > (x-1)(x-3).\end{aligned}$$

Hence the solution set is

$$\{x : \sqrt{4x-3} > x\} = (1, 3).$$

(c) Rearranging this inequality we have

$$\begin{aligned}|17 - 2x^4| \leq 15 &\iff -15 \leq 17 - 2x^4 \leq 15\\ &\iff -32 \leq -2x^4 \leq -2\\ &\iff 16 \geq x^4 \geq 1\\ &\iff 2 \geq |x| \geq 1.\end{aligned}$$

Hence the solution set is

$$\{x : |17 - 2x^4| \leq 15\} = [-2, -1] \cup [1, 2].$$

Solution to Additional Exercise D6

Suppose that $|a| \le 1$. By the backwards form of the Triangle Inequality,

$$\begin{aligned} |a-3| &\ge \big||a|-3\big| \\ &= 3-|a| \quad (\text{since } |a|-3<0) \\ &\ge 2 \qquad\quad\, (\text{since } |a| \le 1). \end{aligned}$$

Hence

$$|a| \le 1 \implies |a-3| \ge 2.$$

Solution to Additional Exercise D7

We rearrange the inequality into a simpler equivalent form:

$$\begin{aligned} &(a^2+b^2)(c^2+d^2) \ge (ac+bd)^2 \\ &\iff a^2c^2+a^2d^2+b^2c^2+b^2d^2 \\ &\ \ge a^2c^2+2acbd+b^2d^2 \\ &\iff a^2d^2+b^2c^2-2acbd \ge 0 \\ &\iff (ad-bc)^2 \ge 0. \end{aligned}$$

The last inequality is true for all a, b, c, $d \in \mathbb{R}$, so the first inequality must also be true for all a, b, c, $d \in \mathbb{R}$, as required.

Solution to Additional Exercise D8

(a) We substitute $x=2$ in the Binomial Theorem and notice that all the terms are positive; this gives, for $n \ge 2$,

$$\begin{aligned} 3^n = (1+2)^n &\ge 1+2n+\frac{n(n-1)}{2!}2^2 \\ &= 1+2n+2n(n-1) \\ &= 1+2n^2. \end{aligned}$$

For $n=1$,

$$3^n = 3 = 1+2n^2.$$

Thus

$$3^n \ge 1+2n^2, \text{ for } n \ge 1.$$

(b) To use mathematical induction, we let $P(n)$ be the statement

$$3^n \ge 1+2n^2.$$

As noted in part (a),

$$3^n = 3 = 1+2n^2, \quad \text{for } n=1,$$

and so $P(1)$ is true.

Now let $k \ge 1$ and assume that $P(k)$ is true; that is,

$$3^k \ge 2k^2+1.$$

We wish to deduce that $P(k+1)$ is true; that is,

$$3^{k+1} \ge 2(k+1)^2+1.$$

Multiplying the inequality $3^k \ge 2k^2+1$ by 3 we get

$$3^{k+1} \ge 3(2k^2+1),$$

so it is sufficient for our purposes to prove that

$$3(2k^2+1) \ge 2(k+1)^2+1, \text{ for } k \ge 1.$$

Now

$$\begin{aligned} &3(2k^2+1) \ge 2(k+1)^2+1 \\ &\iff 6k^2+3 \ge 2k^2+4k+3 \\ &\iff 4k^2 \ge 4k, \end{aligned}$$

and this last equivalent inequality is certainly true for $k \ge 1$. Thus

$$P(k) \implies P(k+1), \text{ for } k \ge 1.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \ge 1$.

Solution to Additional Exercise D9

Since $x = 2/n > -1$ for $n \ge 1$, we can apply Bernoulli's Inequality with $x = 2/n$ to obtain

$$\left(1+\frac{2}{n}\right)^n \ge 1+n\left(\frac{2}{n}\right) = 3.$$

Hence (by Rule 5 with $p=n$) we have

$$1+\frac{2}{n} \ge 3^{1/n}, \text{ for } n \ge 1.$$

Next, since $x = -2/(3n) > -1$ for $n \ge 1$, we can apply Bernoulli's Inequality with $x = -2/(3n)$ to obtain

$$\left(1-\frac{2}{3n}\right)^n \ge 1+n\left(-\frac{2}{3n}\right) = \frac{1}{3}.$$

Hence (by Rule 5 with $p=n$) we have

$$\frac{3n-2}{3n} \ge \left(\frac{1}{3}\right)^{1/n}.$$

We can rewrite this inequality in the form

$$3^{1/n} \ge \frac{3n}{3n-2} = 1+\frac{2}{3n-2}, \text{ for } n \ge 1.$$

Putting these two results together, we get the required inequality.

Solution to Additional Exercise D10

Let $P(n)$ be the statement

$$\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2n)} \le \sqrt{\frac{3/4}{2n+1}}.$$

$P(1)$ is true since

$$\frac{1}{2} = \sqrt{\frac{1}{4}} = \sqrt{\frac{3/4}{2+1}}.$$

Now let $k \ge 1$ and assume that $P(k)$ is true; that is,

$$\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2k-1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2k)} \le \sqrt{\frac{3/4}{2k+1}}.$$

We wish to deduce that $P(k+1)$ is true; that is,

$$\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2(k+1)-1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2(k+1))} \le \sqrt{\frac{3/4}{2(k+1)+1}}$$

which simplifies to

$$\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2k+1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2k+2)} \le \sqrt{\frac{3/4}{2k+3}}.$$

Now

$$\begin{aligned}&\frac{1\cdot 3\cdot 5\cdot \dots \cdot (2k+1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2k+2)}\\ &= \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2k)(2k+2)}\\ &\le \sqrt{\frac{3/4}{2k+1}} \times \frac{2k+1}{2k+2},\end{aligned}$$

using the hypothesis that $P(k)$ is true. Hence, to prove that the statement $P(k+1)$ holds, we need to prove that

$$\sqrt{\frac{3/4}{2k+1}} \times \frac{2k+1}{2k+2} \le \sqrt{\frac{3/4}{2k+3}}.$$

Simplifying this inequality gives

$$\frac{\sqrt{2k+1}}{2k+2} \le \sqrt{\frac{1}{2k+3}}.$$

But

$$\begin{aligned}&\frac{\sqrt{2k+1}}{2k+2} \le \sqrt{\frac{1}{2k+3}}\\ &\iff (2k+1)(2k+3) \le (2k+2)^2\\ &\iff 4k^2+8k+3 \le 4k^2+8k+4.\end{aligned}$$

This last inequality is certainly true. Thus we have shown that

$$P(k) \implies P(k+1), \text{ for } k \ge 1.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \ge 1$.

Solution to Additional Exercise D11

(a) The set E_1 is bounded above by 1, for example, since

$$x \le 1, \quad \text{for } x \in E_1.$$

But E_1 does not have a maximum element. For each $x \in E_1$ we have $0 \le x < 1$, so, by the Density Property, there is a rational number y such that $x < y < 1$. Since $y \in E_1$, we deduce that x is not a maximum element of E_1.

The set E_2 is bounded above by 4, for example, since

$$\left(1+\frac{1}{n}\right)^2 \le (1+1)^2 = 4, \quad \text{for } n = 1, 2, \dots .$$

Also, $4 \in E_2$ since $4 = \left(1+\frac{1}{n}\right)^2$ when $n = 1$ and so 4 is the maximum element of E_2.

(b) The set E_1 is bounded below by 0, for example, since

$$x \ge 0, \quad \text{for } x \in E_1.$$

Also, $0 \in E_1$ and so 0 is the minimum element of E_1.

The set E_2 is bounded below by 1, for example, since

$$\left(1+\frac{1}{n}\right)^2 \ge 1, \quad \text{for } n = 1, 2, \dots .$$

But E_2 does not have a minimum element. For each x in E_2, we have $x = (1+1/n)^2$, for some n, so there is another element of E_2, for example $(1+1/(n+1))^2$, such that

$$\left(1+\frac{1}{n+1}\right)^2 < \left(1+\frac{1}{n}\right)^2 = x.$$

Hence x is not a minimum element of E_2.

(c) As noted in part (a), 1 is an upper bound of E_1. To show that $M = 1$ is the *least* upper bound, we prove that, if $M' < 1$, then there is an element x in E_1 which is greater than M'. Suppose that $M' < 1$. Then, by the Density Property, there is a rational number x such that

$$M' < x < 1 \text{ and } x \ge 0,$$

so $x \in E_1$ and $x > M'$. Thus M' is not an upper bound of E_1. Hence the least upper bound of E_1 is 1.

The least upper bound of E_2 is 4, since $\max E_2 = 4$.

(d) The greatest lower bound of E_1 is 0, since $\min E_1 = 0$.

As noted in part (b), 1 is a lower bound of E_2. To see that $m = 1$ is the *greatest* lower bound of E_2, we have to prove that, if $m' > 1$, then there is an element $(1 + 1/n)^2$ of E_2 such that $(1 + 1/n)^2 < m'$. Suppose that $m' > 1$.

Since $\sqrt{m'} > 1$, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^2 < m' &\iff 1+\frac{1}{n} < \sqrt{m'}\\ &\iff \frac{1}{n} < \sqrt{m'}-1\\ &\iff n > 1/(\sqrt{m'}-1).\end{aligned}$$

By the Archimedean Property, we can choose a positive integer n so large that $n > 1/(\sqrt{m'} - 1)$. Hence, for this value of n, we have

$$\left(1+\frac{1}{n}\right)^2 < m'$$

and so m' is not a lower bound of E_2. Hence the greatest lower bound of E_2 is 1.

Additional exercises for Unit D2

Section 1

Additional Exercise D12

Calculate the first five terms of each of the following sequences (a_n) and draw a sequence diagram in each case.

(a) $a_n = n^2 - 4n + 4, \quad n = 1, 2, \ldots$

(b) $a_n = (-1)^{n+1}/n!, \quad n = 1, 2, \ldots$

(c) $a_n = \sin(\frac{1}{4}n\pi), \quad n = 1, 2, \ldots$

Additional Exercise D13

Determine which of the following sequences (a_n) are monotonic.

(a) $a_n = \dfrac{n}{n+1}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{(-1)^n}{n}, \quad n = 1, 2, \ldots$

(c) $a_n = 2^{1/n}, \quad n = 1, 2, \ldots$

Additional Exercise D14

Prove that the following sequences (a_n) are each eventually monotonic.

(a) $a_n = 5^n/n!, \quad n = 1, 2, \ldots$

(b) $a_n = n + 8/n, \quad n = 1, 2, \ldots$

Section 2

Additional Exercise D15

For each of the following sequences (a_n) and positive numbers ε, find an integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

(a) $a_n = (-1)^n/n^5, \quad \varepsilon = 0.001$

(b) $a_n = 1/(2n+1)^2, \quad \varepsilon = 0.002$

Additional Exercise D16

Use the definition of a null sequence to prove that the two sequences in Additional Exercise D15 are null.

Additional Exercise D17

Prove that the following sequences (a_n) are not null.

(a) $a_n = \sqrt{n}, \quad n = 1, 2, \ldots$

(b) $a_n = 1 + \dfrac{(-1)^n}{n}, \quad n = 1, 2, \ldots$

Additional Exercise D18

Use the fact that $(1/n)$ is null to deduce that the following sequences (a_n) are null. State which rules you use.

(a) $a_n = \dfrac{2}{\sqrt{n}} + \dfrac{3}{n^7}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{\cos n}{n^2+1}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{n!}{n^n}, \quad n = 1, 2, \ldots$

Additional Exercise D19

Use the identity $a - b = (a^2 - b^2)/(a + b)$ to prove that the sequence (a_n) is null, where

$$a_n = \sqrt{n+1} - \sqrt{n}, \quad n = 1, 2, \ldots.$$

Additional Exercise D20

Use the list of basic null sequences to prove that the following sequences (a_n) are null. State which rules you use.

(a) $a_n = \dfrac{3}{4^n} + \dfrac{2n}{3^n}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{6n^{10}}{n!}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{n^{10}10^n}{n!}, \quad n = 1, 2, \ldots$

Hint: In part (c), you can express the sequence as a product of basic null sequences.

Additional Exercise D21 Challenging

Determine whether or not the sequence (a_n) is null, where

$$a_n = (n+1)^{3/2} - n^{3/2}, \quad n = 1, 2, \ldots.$$

Hint: Use the same identity as in Additional Exercise D19.

Section 3

Additional Exercise D22

Show that the following sequences (a_n) converge to 1, by calculating $a_n - 1$ in each case.

(a) $a_n = \dfrac{n-1}{n+3}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{n^2}{n^2+n+1}, \quad n = 1, 2, \ldots$

Additional Exercise D23

Use the Combination Rules to find the limits of the following sequences (a_n).

(a) $a_n = \dfrac{n^2}{n^2+n+1}, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{n^2 - 2^n}{2^n + n^{20}}, \quad n = 1, 2, \ldots$

(c) $a_n = \dfrac{5n! + 5^n}{n^{100} + n!}, \quad n = 1, 2, \ldots$

Additional Exercise D24

Prove that if (a_n) is convergent with limit l, then (a_n^2) is convergent with limit l^2.

Section 4

Additional Exercise D25

Classify the following sequences (a_n) as bounded or unbounded and as convergent or divergent.

(a) $a_n = n^{1/4}, \quad n = 1, 2, \ldots$

(b) $a_n = 100^n/n!, \quad n = 1, 2, \ldots$

Additional Exercise D26

Use the Reciprocal Rule to prove that the following sequences (a_n) tend to infinity.

(a) $a_n = n!/n^3, \quad n = 1, 2, \ldots$

(b) $a_n = n^2 + 2n, \quad n = 1, 2, \ldots$

(c) $a_n = n^2 - 2n, \quad n = 1, 2, \ldots$

(d) $a_n = n! - n^3 - 3^n, \quad n = 1, 2, \ldots$

Additional Exercise D27

Use the Subsequence Rules to prove that the following sequences (a_n) are divergent.

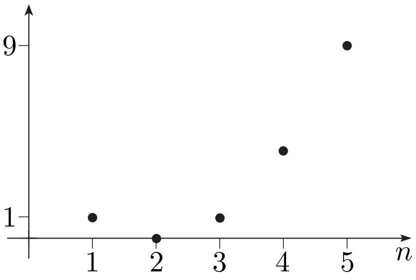
(a) $a_n = (-1)^n 2^n, \quad n = 1, 2, \ldots$

(b) $a_n = \dfrac{(-1)^n n^2}{2n^2+1}, \quad n = 1, 2, \ldots$

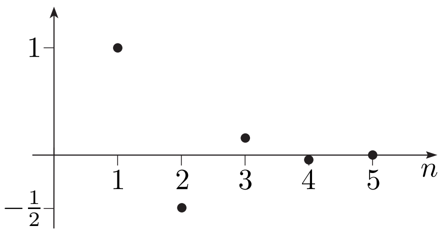
Solutions to additional exercises for Unit D2

Solution to Additional Exercise D12

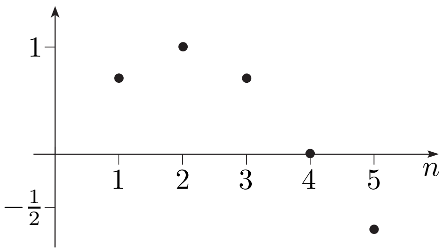
(a) $1, 0, 1, 4, 9.$



(b) $1, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, \frac{1}{120}.$



(c) $\dfrac{1}{\sqrt{2}}, 1, \dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}.$



Solution to Additional Exercise D13

(a) Since $a_n > 0$ for all n, we can use Strategy D4. We have

$$a_n = \frac{n}{n+1} \text{ and } a_{n+1} = \frac{n+1}{n+2}$$

so

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)/(n+2)}{n/(n+1)} \\ &= \frac{n^2+2n+1}{n^2+2n} \geq 1, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Thus (a_n) is increasing, so (a_n) is monotonic.

(b) We have

$$a_1 = -1, \; a_2 = \tfrac{1}{2}, \; a_3 = -\tfrac{1}{3}.$$

Since $a_1 < a_2$, we know that (a_n) is not decreasing; and since $a_3 < a_2$, we know that (a_n) is not increasing.

Hence (a_n) is not monotonic.

(c) We have

$$a_{n+1} = 2^{1/(n+1)} \leq 2^{1/n} = a_n, \quad \text{for } n = 1, 2, \ldots.$$

This holds because, by the rules for inequalities (Rule 5 with $p = n(n+1)$),

$$2^{1/(n+1)} \leq 2^{1/n} \iff 2^n \leq 2^{n+1},$$

and the right-hand inequality is true for $n = 1, 2, \ldots$.

Thus (a_n) is decreasing, so (a_n) is monotonic.

Solution to Additional Exercise D14

(a) We have $a_n > 0$ for all n and

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}/(n+1)!}{5^n/n!} = \frac{5}{n+1}.$$

Now

$$\frac{5}{n+1} \leq 1, \quad \text{for } n = 4, 5, 6, \ldots,$$

so

$$a_{n+1} \leq a_n, \quad \text{for } n = 4, 5, 6, \ldots.$$

Hence (a_n) is eventually decreasing, so (a_n) is eventually monotonic.

(b) We have

$$\begin{aligned}a_{n+1} - a_n &= \left(n+1+\frac{8}{n+1}\right) - \left(n + \frac{8}{n}\right) \\ &= 1 - \frac{8}{n(n+1)}.\end{aligned}$$

Now

$$\begin{aligned}1 - \frac{8}{n(n+1)} \geq 0 &\iff \frac{8}{n(n+1)} \leq 1 \\ &\iff n(n+1) \geq 8\end{aligned}$$

and this final inequality holds for $n \geq 3$. So

$$a_{n+1} \geq a_n, \quad \text{for } n \geq 3.$$

Hence (a_n) is eventually increasing, so (a_n) is eventually monotonic.

Solution to Additional Exercise D15

(a) We have $|a_n| = \left|\dfrac{(-1)^n}{n^5}\right| = \dfrac{1}{n^5}$, and

$$\begin{aligned}\frac{1}{n^5} < 0.001 &\iff n^5 > 1000 \\ &\iff n > 1000^{1/5} \approx 3.98.\end{aligned}$$

Hence

$$|a_n| < 0.001, \quad \text{for all } n > 3,$$

so we can take $N = 3$.

(b) We have $|a_n| = 1/(2n+1)^2$, and

$$\begin{aligned}\frac{1}{(2n+1)^2} < 0.002 &\iff (2n+1)^2 > 500\\ &\iff n > \tfrac{1}{2}(\sqrt{500}-1) \approx 10.68.\end{aligned}$$

Hence

$$|a_n| < 0.002, \quad \text{for all } n > 10,$$

so we can take $N = 10$.

Solution to Additional Exercise D16

(a) We want to show that:

for each $\varepsilon > 0$, there is an integer N such that

$$\left|\frac{(-1)^n}{n^5}\right| = \frac{1}{n^5} < \varepsilon, \quad \text{for all } n > N. \qquad \text{(S1)}$$

We know that

$$\begin{aligned}\frac{1}{n^5} < \varepsilon &\iff n^5 > \frac{1}{\varepsilon}\\ &\iff n > \left(\frac{1}{\varepsilon}\right)^{1/5}.\end{aligned}$$

If we take $N \geq (1/\varepsilon)^{1/5}$, then statement (S1) holds. Hence (a_n) is null.

(b) We want to show that:

for each $\varepsilon > 0$, there is an integer N such that

$$\frac{1}{(2n+1)^2} < \varepsilon, \quad \text{for all } n > N. \qquad \text{(S2)}$$

We know that

$$\begin{aligned}\frac{1}{(2n+1)^2} < \varepsilon &\iff (2n+1)^2 > 1/\varepsilon\\ &\iff n > \tfrac{1}{2}\left(\sqrt{1/\varepsilon}-1\right).\end{aligned}$$

If we take $N \geq \frac{1}{2}(\sqrt{1/\varepsilon}-1)$, then statement (S2) holds. Hence (a_n) is null.

Solution to Additional Exercise D17

(a) We use Strategy D5. We have to find a positive number ε for which there is no integer N such that

$$|a_n| = \sqrt{n} < \varepsilon, \quad \text{for all } n > N.$$

Since $\sqrt{n} > \frac{1}{2}$ for all n, we can take $\varepsilon = \frac{1}{2}$, for example. Hence (a_n) is not null.

(b) We use Strategy D5. We have to find a positive number ε for which there is no integer N such that

$$|a_n| = \left|1 + \frac{(-1)^n}{n}\right| < \varepsilon, \quad \text{for all } n > N.$$

Since

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{2} \quad \text{for } n \geq 2,$$

we have

$$-\frac{1}{2} \leq \frac{(-1)^n}{n} \leq \frac{1}{2}, \quad \text{for } n \geq 2,$$

so

$$\frac{1}{2} \leq 1 + \frac{(-1)^n}{n} \leq \frac{3}{2}, \quad \text{for } n \geq 2.$$

So we can take $\varepsilon = \frac{1}{2}$, for example. Hence (a_n) is not null.

Solution to Additional Exercise D18

(a) The sequences $\left(\dfrac{2}{\sqrt{n}}\right)$ and $\left(\dfrac{3}{n^7}\right)$ are null,by the Power Rule and the Multiple Rule.

Hence (a_n) is null, by the Sum Rule.

(b) We guess that (a_n) is dominated by (b_n), where

$$b_n = \frac{1}{n}, \quad n = 1, 2, \ldots.$$

This holds because

$$|\cos n| \leq 1 \quad \text{for } n = 1, 2, \ldots,$$

and

$$n^2 + 1 \geq n \quad \text{for } n = 1, 2, \ldots,$$

so that

$$\left|\frac{\cos n}{n^2+1}\right| \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Since (b_n) is null, we deduce that (a_n) is null, by the Squeeze Rule.

(c) Since

$$\frac{n!}{n^n} = \frac{1 \times 2 \times \cdots \times n}{n \times n \times \cdots \times n} = \frac{1}{n} \times \frac{2}{n} \times \cdots \times \frac{n-1}{n} \times \frac{n}{n},$$

we have

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Hence (a_n) is null, by the Squeeze Rule.

Solution to Additional Exercise D19

Using the given identity, we obtain

$$\sqrt{n+1}-\sqrt{n}=\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Thus, for $n=1,2,\dots$, we have

$$0\leq\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{\sqrt{n}}.$$

Now the sequence $\left(\frac{1}{\sqrt{n}}\right)$ is null, by the Power Rule.

Hence (a_n) is null, by the Squeeze Rule.

Solution to Additional Exercise D20

(a) Since

$$\frac{1}{4^n}=\left(\frac{1}{4}\right)^n \quad\text{and}\quad \frac{n}{3^n}=n\left(\frac{1}{3}\right)^n,$$

both $(1/4^n)$ and $(n/3^n)$ are basic null sequences, so (a_n) is null, by the Multiple Rule and the Sum Rule.

(b) $(n^{10}/n!)$ is a basic null sequence, so (a_n) is a null sequence, by the Multiple Rule.

(c) First note that

$$\frac{n^{10}10^n}{n!}=\frac{n^{10}}{10^n}\,\frac{100^n}{n!},\qquad\text{for } n=1,2,\dots.$$

Since $(n^{10}/10^n)$ and $(100^n/n!)$ are basic null sequences, (a_n) is null, by the Product Rule.

Solution to Additional Exercise D21

The sequence (a_n) is not null.

To prove this, we use Strategy D5. We have to find a positive number ε for which there is no integer N such that

$$|a_n|<\varepsilon,\qquad\text{for all } n>N.$$

Using the same identity as in Additional Exercise D19, we have that

$$\begin{aligned}a_n=(n+1)^{3/2}-n^{3/2}&=\frac{(n+1)^3-n^3}{(n+1)^{3/2}+n^{3/2}}\\&=\frac{3n^2+3n+1}{(n+1)^{3/2}+n^{3/2}}\\&>\frac{3n^2}{(n+1)^{3/2}+n^{3/2}}\\&\geq\frac{3n^2}{(2n)^{3/2}+n^{3/2}}\\&=\frac{3n^2}{(2^{3/2}+1)n^{3/2}}>\frac{3}{4}\sqrt{n}.\end{aligned}$$

In particular, $a_n>\frac{3}{4}\sqrt{n}>\frac{3}{4}$, for $n=1,2,\dots$.

So we can take $\varepsilon=\frac{3}{4}$, for example.

Solution to Additional Exercise D22

(a) We have

$$a_n-1=\frac{n-1}{n+3}-1=\frac{-4}{n+3}.$$

Now $\left(\frac{-4}{n+3}\right)$ is a null sequence, by the Squeeze Rule, since

$$\left|\frac{-4}{n+3}\right|=\frac{4}{n+3}\leq\frac{4}{n},\qquad\text{for } n=1,2,\dots.$$

Thus (a_n) is convergent with limit 1.

(b) We have

$$a_n-1=\frac{n^2}{n^2+n+1}-1=-\frac{n+1}{n^2+n+1}.$$

Now $\left(-\frac{n+1}{n^2+n+1}\right)$ is a null sequence, by the Squeeze Rule, since

$$\left|-\frac{n+1}{n^2+n+1}\right|\frac{n+1}{n^2+n+1}=\frac{n+1}{n(n+1)+1}\leq\frac{1}{n},$$

for $n=1,2,\dots$. Thus (a_n) is convergent with limit 1.

Solution to Additional Exercise D23

In each case we apply Strategy D7.

(a) The dominant term is n^2, so we write

$$a_n=\frac{n^2}{n^2+n+1}=\frac{1}{1+1/n+1/n^2}.$$

Since $(1/n)$ and $(1/n^2)$ are basic null sequences, we find, by the Combination Rules, that

$$\lim_{n\to\infty}a_n=\frac{1}{1+0+0}=1.$$

(b) The dominant term is 2^n, so we write

$$\frac{n^2 - 2^n}{2^n + n^{20}} = \frac{n^2/2^n - 1}{1 + n^{20}/2^n}.$$

Since $(n^2/2^n) = (n^2(\frac{1}{2})^n)$ and $(n^{20}/2^n) = (n^{20}(\frac{1}{2})^n)$ are basic null sequences, we find, by the Combination Rules, that

$$\lim_{n\to\infty} a_n = \frac{0-1}{1+0} = -1.$$

(c) The dominant term is $n!$, so we write

$$\frac{5n! + 5^n}{n^{100} + n!} = \frac{5 + 5^n/n!}{n^{100}/n! + 1}.$$

Since $(5^n/n!)$ and $(n^{100}/n!)$ are basic null sequences, we find, by the Combination Rules, that

$$\lim_{n\to\infty} a_n = \frac{5+0}{0+1} = 5.$$

Solution to Additional Exercise D24

The result follows from the Product Rule with $b_n = a_n$, for $n = 1, 2, \ldots$.

Solution to Additional Exercise D25

(a) This sequence is unbounded and hence divergent, by Corollary D15.

(b) This sequence is a basic null sequence. Hence it is convergent and therefore bounded, by Theorem D14.

Solution to Additional Exercise D26

(a) Each term of the sequence (a_n) is positive. Also

$$\frac{1}{a_n} = \frac{n^3}{n!}$$

and $(n^3/n!)$ is a basic null sequence.

Hence $a_n \to \infty$, by the Reciprocal Rule.

(b) Each term of the sequence (a_n) is positive and

$$0 \le \frac{1}{a_n} = \frac{1}{n^2+2n} \le \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since $(1/n^2)$ is null, we deduce that $(1/a_n)$ is null, by the Squeeze Rule.

Hence $a_n \to \infty$, by the Reciprocal Rule.

(c) The dominant term is n^2, so we first write

$$a_n = n^2 - 2n = n^2(1 - 2/n), \quad \text{for } n = 1, 2, \ldots.$$

Since $2/n < 1$, for $n > 2$, we deduce that (a_n) is eventually positive.

Next we write

$$\frac{1}{a_n} = \frac{1}{n^2 - 2n} = \frac{1/n^2}{1 - 2/n}.$$

Now $(1/n)$ and $(1/n^2)$ are basic null sequences. Thus, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0}{1-0} = 0.$$

Hence $a_n \to \infty$, by the Reciprocal Rule.

(d) The dominant term is $n!$, so we first write

$$a_n = n! - n^3 - 3^n = n!\left(1 - \frac{n^3}{n!} - \frac{3^n}{n!}\right),$$

for $n = 1, 2, \ldots$.

Since $(n^3/n!)$ and $(3^n/n!)$ are basic null sequences, we know that there is some integer N such that

$$\frac{n^3}{n!} + \frac{3^n}{n!} < 1, \quad \text{for } n > N.$$

Thus (a_n) is eventually positive.

Next we write

$$\frac{1}{a_n} = \frac{1}{n! - n^3 - 3^n} = \frac{1/n!}{1 - n^3/n! - 3^n/n!}.$$

Now $(1/n!)$, $(n^3/n!)$ and $(3^n/n!)$ are basic null sequences. Thus, by the Combination Rules,

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{0}{1-0-0} = 0.$$

Hence $a_n \to \infty$, by the Reciprocal Rule.

Solution to Additional Exercise D27

(a) We have

$$a_{2k} = 2^{2k} = 4^k, \quad \text{for } k = 1, 2, \ldots.$$

Each term of (a_{2k}) is positive. Also

$$\frac{1}{a_{2k}} = \frac{1}{4^k} = \left(\frac{1}{4}\right)^k.$$

and $((1/4)^k)$ is a basic null sequence. Hence $a_{2k} \to \infty$, by the Reciprocal Rule. Thus (a_n) is divergent, by the Second Subsequence Rule.

(b) We have

$$a_{2k} = \frac{(2k)^2}{2(2k)^2+1} = \frac{4k^2}{8k^2+1} = \frac{4}{8+1/k^2}$$

and

$$a_{2k-1} = \frac{-(2k-1)^2}{2(2k-1)^2+1} = -\frac{4k^2-4k+1}{8k^2-8k+3}$$
$$= -\frac{4 - 4/k + 1/k^2}{8 - 8/k + 3/k^2}.$$

Now $(1/k)$ and $(1/k^2)$ are basic null sequences, so

$$\lim_{k\to\infty} a_{2k} = \tfrac{1}{2}, \quad \text{whereas} \quad \lim_{k\to\infty} a_{2k-1} = -\tfrac{1}{2},$$

by the Combination Rules.

Hence (a_n) is divergent, by the First Subsequence Rule.

Additional exercises for Unit D3

Section 1

Additional Exercise D28

Prove that $\displaystyle\sum_{n=0}^{\infty}\left(-\tfrac{4}{5}\right)^n$ is convergent and find its sum.

Additional Exercise D29

Prove that

$$\frac{1}{2n-1}-\frac{1}{2n+1}=\frac{2}{4n^2-1},\quad \text{for } n=1,2,\dots,$$

and deduce that

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\tfrac{1}{2}.$$

Additional Exercise D30

Determine the behaviour of the following series. If the series is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\left(\frac{4}{5}\right)^n+\frac{4}{n(n+2)}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty}\left(1+\left(\tfrac{1}{2}\right)^n\right)$

Hint: For part (a), you may find it helpful to refer to the solution to Exercise D43 in Unit D3.

Section 2

Additional Exercise D31

Determine whether or not the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(1/n)}{2n^2+3}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2n^3-n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n}}{4n^3+n+2}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^5}{2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 3^n}{n!}$ (f) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$

Additional Exercise D32 Challenging

(a) Use the Ratio Test to show that

$$\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}\ \text{converges}$$

but

$$\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}\ \text{diverges.}$$

(b) For which positive values of c can you use the Ratio Test to prove that $\displaystyle\sum_{n=1}^{\infty}\frac{c^n n!}{n^n}$ is convergent?

Hint: Use the fact (from Unit D2) that $(1+1/n)^n\to e$ as $n\to\infty$.

Section 3

Additional Exercise D33

Determine which of the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{1+\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{n^4+3}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{n+2^n}{3^n+5}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+2}$

Additional Exercise D34 Challenging

Prove that

$$\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}>\frac{1}{3n},\quad \text{for } n=1,2,\dots,$$

and deduce that the series

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots \qquad (1)$$

is divergent.

Solutions to additional exercises for Unit D3

Solution to Additional Exercise D28

This is a geometric series with $a = 1$ and $r = -\frac{4}{5}$. Since $|r| = |-\frac{4}{5}| = \frac{4}{5} < 1$, the series is convergent, with sum

$$\frac{1}{1-(-\frac{4}{5})} = \frac{5}{9}.$$

Solution to Additional Exercise D29

For $n = 1, 2, \ldots,$

$$\begin{aligned}\frac{1}{2n-1} - \frac{1}{2n+1} &= \frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)} \\ &= \frac{2}{(2n-1)(2n+1)} = \frac{2}{4n^2-1},\end{aligned}$$

as required.

Thus the given series is a telescoping series and the nth partial sum s_n is given by

$$\begin{aligned}s_n &= \sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \right. \\ &\qquad \left. + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right).\end{aligned}$$

All the terms cancel out except for the first term in the first bracket and the second term in the last bracket. Thus

$$s_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).$$

Since $(1/(2n+1))$ is a null sequence, it follows that

$$\lim_{n\to\infty} s_n = \tfrac{1}{2},$$

so the series is convergent, with sum $\frac{1}{2}$.

Solution to Additional Exercise D30

(a) The series $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{4}{5}\right)^n$ is a geometric series, with $a = r = \frac{4}{5}$. Since $|r| = \frac{4}{5} < 1$, the series is convergent, with sum

$$\frac{\frac{4}{5}}{1-\frac{4}{5}} = 4.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent, with sum $\frac{3}{4}$; see the solution to Exercise D43 in Unit D3.

Hence, by the Combination Rules,

$$\sum_{n=1}^{\infty}\left(\left(\tfrac{4}{5}\right)^n + \frac{4}{n(n+2)}\right) \text{ is convergent,}$$

with sum $4 + \left(4 \times \frac{3}{4}\right) = 7$.

(b) Let $a_n = 1 + (\frac{1}{2})^n$. Since $((\frac{1}{2})^n)$ is a basic null sequence, it follows from the Combination Rules for sequences that

$$\lim_{n\to\infty} a_n = 1 \neq 0.$$

Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty}\left(1 + \left(\tfrac{1}{2}\right)^n\right) \text{ is divergent.}$$

Solution to Additional Exercise D31

(a) We guess that this series is dominated by $\sum(1/n^2)$. Indeed, since

$$0 \leq 1/n \leq \frac{\pi}{2}, \quad \text{for } n = 1, 2, \ldots,$$

we have

$$0 \leq \cos(1/n) \leq 1, \quad \text{for } n = 1, 2, \ldots.$$

Hence, for $n = 1, 2, \ldots,$

$$0 \leq \frac{\cos(1/n)}{2n^2+3} \leq \frac{1}{2n^2+3} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ is a basic convergent series, we deduce by the Comparison Test that

$$\sum_{n=1}^{\infty} \frac{\cos(1/n)}{2n^2+3} \text{ is convergent.}$$

(b) Let

$$a_n = \frac{n^2}{2n^3-n}, \quad \text{for } n = 1, 2, \ldots.$$

We guess that, for large n, the terms of this series behave like $n^2/n^3 = 1/n$ and we know that $\sum_{n=1}^{\infty} 1/n$ is divergent. We cannot compare a_n and $1/n$ directly so we use the Limit Comparison Test with

$$b_n = \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive and, by the Combination Rules for sequences,

$$\begin{aligned}\frac{a_n}{b_n} &= \left(\frac{n^2}{2n^3-n}\right)\left(\frac{n}{1}\right)\\ &= \frac{1}{2-1/n^2} \to \tfrac{1}{2} \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty} 1/n$ is a basic divergent series, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{n^2}{2n^3-n} \text{ is divergent.}$$

(c) Let

$$a_n = \frac{\sqrt{2n}}{4n^3+n+2}, \quad \text{for } n = 1, 2, \ldots .$$

We guess that the terms of this series behave like $\sqrt{n}/n^3 = 1/n^{5/2}$ and we know that $\sum_{n=1}^{\infty} 1/n^{5/2}$ is convergent. We cannot compare a_n and $1/n^{5/2}$ directly so we use the Limit Comparison Test with

$$b_n = \frac{1}{n^{5/2}}, \quad \text{for } n = 1, 2, \ldots .$$

Both a_n and b_n are positive and, by the Combination Rules for sequences,

$$\begin{aligned}\frac{a_n}{b_n} &= \left(\frac{\sqrt{2n}}{4n^3+n+2}\right)\left(\frac{n^{5/2}}{1}\right)\\ &= \frac{\sqrt{2}\,n^3}{4n^3+n+2}\\ &= \frac{\sqrt{2}}{4+1/n^2+2/n^3} \to \tfrac{1}{4}\sqrt{2} \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty} 1/n^{5/2}$ is a basic convergent series, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n}}{4n^3+n+2} \text{ is convergent.}$$

(d) Let

$$a_n = \frac{(n+1)^5}{2^n}, \quad \text{for } n = 1, 2, \ldots .$$

We guess that the terms of this series behave like $n^5/2^n$ and we know that $\sum_{n=1}^{\infty} n^5/2^n$ is convergent, so we use the Limit Comparison Test with

$$b_n = \frac{n^5}{2^n}, \quad n = 1, 2, \ldots .$$

Both a_n and b_n are positive and, by the Combination Rules for sequences,

$$\begin{aligned}\frac{a_n}{b_n} &= \left(\frac{(n+1)^5}{2^n}\right)\left(\frac{2^n}{n^5}\right)\\ &= \left(\frac{n+1}{n}\right)^5 = \left(1+\frac{1}{n}\right)^5 \to 1 \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty} n^5/2^n$ is a basic convergent series, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{(n+1)^5}{2^n} \text{ is convergent.}$$

(Alternatively, we can either note that

$$\sum_{n=1}^{\infty} \frac{(n+1)^5}{2^n} = \sum_{n=2}^{\infty} \frac{n^5}{2^{n-1}} = 2\sum_{n=2}^{\infty} \frac{n^5}{2^n},$$

and use the Multiple Rule, or we can use the Ratio Test:

$$\frac{a_{n+1}}{a_n} = \left(\frac{(n+2)^5}{2^{n+1}}\right)\left(\frac{2^n}{(n+1)^5}\right) = \frac{1}{2}\left(\frac{1+2/n}{1+1/n}\right)^5,$$

which converges to $\frac{1}{2} < 1$.)

(e) There is no obvious basic series to compare with, so we try the Ratio Test. Let

$$a_n = \frac{n^2 3^n}{n!}, \quad \text{for } n = 1, 2, \ldots .$$

Then a_n is positive and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{(n+1)^2 3^{n+1}}{(n+1)!}\right)\left(\frac{n!}{n^2 3^n}\right)\\ &= \frac{3(n+1)}{n^2}\\ &= 3\left(\frac{1}{n}+\frac{1}{n^2}\right).\end{aligned}$$

Hence, by the Combination Rules for sequences,

$$\frac{a_{n+1}}{a_n} \to 0 < 1, \quad \text{as } n \to \infty.$$

Thus, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{n^2 3^n}{n!} \text{ is convergent.}$$

(f) Again, we try the Ratio Test. Let

$$a_n = \frac{(n!)^2}{(2n)!}, \quad n = 1, 2, \ldots .$$

Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{((n+1)!)^2}{(2n+2)!}\right)\left(\frac{(2n)!}{(n!)^2}\right)\\ &= \frac{(n+1)^2}{(2n+2)(2n+1)}\\ &= \frac{n+1}{2(2n+1)} = \frac{1+1/n}{2(2+1/n)}.\end{aligned}$$

Hence, by the Combination Rules for sequences,

$$\frac{a_{n+1}}{a_n} \to \frac{1}{4} < 1 \quad \text{as } n \to \infty.$$

Thus, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \text{ is convergent.}$$

Solution to Additional Exercise D32

(a) Let

$$a_n = \frac{2^n n!}{n^n}, \quad \text{for } n = 1, 2, \ldots;$$

then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\right)\left(\frac{n^n}{2^n n!}\right)\\ &= \frac{2(n+1)\,n^n}{(n+1)^{n+1}}\\ &= \frac{2n^n}{(n+1)^n} = \frac{2}{(1+1/n)^n}.\end{aligned}$$

Thus

$$\frac{a_{n+1}}{a_n} \to \frac{2}{e} < 1, \quad \text{as } n \to \infty.$$

Thus, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \text{ is convergent.}$$

If $a_n = 3^n n!/n^n$, then a similar calculation yields

$$\frac{a_{n+1}}{a_n} \to \frac{3}{e} > 1, \quad \text{as } n \to \infty.$$

so, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \text{ is divergent.}$$

(b) We follow the same approach as in part (a). So, for any positive number c, let

$$a_n = \frac{c^n n!}{n^n}, \quad \text{for } n = 1, 2, \ldots;$$

then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{c^{n+1}(n+1)!}{(n+1)^{n+1}}\right) \times \left(\frac{n^n}{c^n n!}\right)\\ &= \frac{c(n+1)n^n}{(n+1)^{n+1}}\\ &= \frac{cn^n}{(n+1)^n} = \frac{c}{(1+1/n)^n}.\end{aligned}$$

Hence

$$\frac{a_{n+1}}{a_n} \to \frac{c}{e}, \quad \text{as } n \to \infty.$$

It follows, by the Ratio Test, that the series $\sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$ is convergent if $0 < c < e$ and divergent if $c > e$. However, if $c = e$ the ratio

$$\frac{a_{n+1}}{a_n} \to 1, \quad \text{as } n \to \infty;$$

and so the Ratio Test gives us no conclusion either way.

Remark If $c = e$, then the series is also divergent, but this is harder to prove. One method is to use the inequality

$$n! > \left(\frac{n+1}{e}\right)^n, \quad \text{for } n = 1, 2, \ldots.$$

From this inequality, we deduce that

$$\frac{e^n n!}{n^n} > \left(\frac{n+1}{n}\right)^n = (1+1/n)^n, \quad \text{for } n = 1, 2, \ldots.$$

Since $\lim_{n\to\infty} (1+1/n)^n = e \neq 0$, we deduce that $(e^n n!/n^n)$ is not a null sequence.

Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n} \text{ is divergent.}$$

Solution to Additional Exercise D33

(a) We use the Alternating Test.

Let

$$a_n = \frac{(-1)^{n+1}}{1+\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$ where

$$b_n = \frac{1}{1+\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a null sequence by the Squeeze Rule, since
$$0 \leq b_n = \frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots$$
3. (b_n) is decreasing, because $(1/b_n) = (1+\sqrt{n})$ is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt{n}} \text{ is convergent.}$$

(b) We guess that this series is dominated by the basic convergent series $\sum(1/n^2)$. Indeed, let

$$a_n = \frac{\sin n}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Then

$$|a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}, \text{ for } n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}(1/n^2)$ is convergent, it follows from the Comparison Test that $\sum_{n=1}^{\infty}|a_n|$ is convergent. Hence, by the Absolute Convergence Test,

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2} \text{ is convergent.}$$

(c) Let

$$a_n = \frac{(-1)^{n+1}n!}{n^4+3}, \quad \text{for } n = 1, 2, \ldots.$$

Then

$$\frac{1}{|a_n|} = \frac{n^4+3}{n!} = \frac{n^4}{n!} + \frac{3}{n!} \to 0.$$

It follows from the Reciprocal Rule that $|a_n| \to \infty$ as $n \to \infty$. Thus, by the Non-null Test,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n!}{n^4+3} \text{ is divergent.}$$

(d) Let

$$a_n = \frac{n+2^n}{3^n+5}, \quad \text{for } n = 1, 2, \ldots.$$

We guess that a_n behaves like $2^n/3^n = \left(\frac{2}{3}\right)^n$ for large n, so we use the Limit Comparison Test with

$$b_n = \left(\tfrac{2}{3}\right)^n, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive and

$$\begin{aligned}\frac{a_n}{b_n} &= \left(\frac{n+2^n}{3^n+5}\right)\left(\frac{3}{2}\right)^n \\ &= \frac{3^n(n+2^n)}{2^n(3^n+5)} \\ &= \frac{n/2^n+1}{1+5/3^n} \to 1 \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is a basic convergent series, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty}\frac{n+2^n}{3^n+5} \text{ is convergent.}$$

(e) We use the Alternating Test.

Let

$$a_n = \frac{(-1)^{n+1}n}{n^2+2}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1}b_n$, where

$$b_n = \frac{n}{n^2+2}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. $b_n = \dfrac{n}{n^2+2} = \dfrac{1/n}{1+2/n^2} \to 0$ as $n \to \infty$ so (b_n) is a null sequence
3. (b_n) is decreasing, because $$(1/b_n) = \left(\frac{n^2+2}{n}\right) = \left(n + \frac{2}{n}\right) \text{ is increasing.}$$

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+2} \text{ is convergent.}$$

Solution to Additional Exercise D34

Since $3n-2 < 3n$ and $3n-1 < 3n$, we have

$$\begin{aligned}\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} &> \frac{1}{3n} + \frac{1}{3n} - \frac{1}{3n} \\ &= \frac{1}{3n}, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

We now write series (1) as $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}, \text{ for } n = 1, 2, \ldots.$$

We have just shown that

$$a_n > \frac{1}{3n}, \text{ for } n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}(1/n)$ is a basic divergent series, it follows from the Corollary to the Multiple Rule (Corollary D26) that $\sum_{n=1}^{\infty}(1/(3n))$ is also divergent. It now follows from the Comparison Test that

$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Additional exercises for Unit D4

Section 1

Additional Exercise D35

Let f and g be the functions

$$f(x) = \tan x \quad (x \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi))$$

and

$$g(x) = \sqrt{x} \quad (x \in [0, \infty)).$$

Determine the domain and the rule of the following functions.

(a) $f + g$ (b) fg (c) f/g (d) $f \circ g$

(e) $g \circ f$

Additional Exercise D36

Prove that the following function has an inverse function, and find the domain and rule of this inverse function.

$$f(x) = \frac{x-1}{x+2} \quad (x \in (-2, \infty))$$

Hint: It may help to write

$$\frac{x-1}{x+2} = 1 - \frac{3}{x+2}.$$

Additional Exercise D37

By checking the definition, prove that each of the following functions is strictly monotonic.

(a) $f(x) = x^3 + 1 - \dfrac{1}{x^2} \quad (x \in (0, \infty))$

(b) $f(x) = \dfrac{1}{(1+x^3)^2} \quad (x \in [0, \infty))$

Section 2

Additional Exercise D38

Use the appropriate rules, together with the list of basic continuous functions, to prove that the following functions are continuous. (Recall that $\exp(x)$ is an alternative way of writing e^x and is useful when taking the exponential of a long expression.)

(a) $f(x) = \exp(\sin(x^2 + 1)) \quad (x \in \mathbb{R})$

(b) $f(x) = e^{\sqrt{x}} + x^5 \quad (x \in [0, \infty))$

Additional Exercise D39

Determine whether the following functions are continuous at 0.

(a) $f(x) = \begin{cases} x, & x < 0, \\ \dfrac{1}{x+1}, & x \geq 0. \end{cases}$

(b) $f(x) = \begin{cases} \sin x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$

(c) $f(x) = \begin{cases} (1/x)\cos(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$

(d) $f(x) = \begin{cases} 0, & x < 0, \\ \sqrt{x}, & x \geq 0. \end{cases}$

Additional Exercise D40

Prove that the following function is continuous at $\frac{1}{2}\pi$ and at $-\frac{1}{2}\pi$.

$$f(x) = \begin{cases} -1, & x \leq -\frac{1}{2}\pi, \\ \sin x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ 1, & x \geq \frac{1}{2}\pi. \end{cases}$$

Section 3

Additional Exercise D41

Prove that the function

$$f(x) = x - \sin x - \tfrac{2}{3}\pi \quad (x \in \mathbb{R})$$

has a zero in the interval $\left(\tfrac{2}{3}\pi, \tfrac{5}{6}\pi\right)$.

Additional Exercise D42

Suppose that the function $f : [0,1] \longrightarrow [0,1]$ is continuous. Prove that the equation

$$f(x) = x^3$$

has a solution in the interval $[0,1]$.

Hint: Consider the function $g(x) = f(x) - x^3$ $(x \in [0,1])$.

Additional Exercise D43

Prove that each of the following polynomials has the stated number of zeros:

(a) $p(x) = x^4 - 4x^3 + 3x^2 + 2x - 1$, four zeros

(b) $p(x) = 3x^3 - 8x^2 + x + 3$, three zeros.

Section 4

Additional Exercise D44

Prove that each of the following functions has a continuous inverse function and determine the domain of the inverse function in each case.

(a) $f(x) = x^3 + 1 - \dfrac{1}{x^2} \quad (x \in (0, \infty))$

(b) $f(x) = \dfrac{1}{(1+x^3)^2} \quad (x \in [0, \infty))$

Hint: We proved that these functions are monotonic in Additional Exercise D37.

Additional Exercise D45

State whether or not each of the following statements is true.

(a) $\sin(\sin^{-1} x) = x$, for $x \in [-1, 1]$.

(b) $\sin^{-1}(\sin x) = x$, for $x \in \mathbb{R}$.

Additional Exercise D46

(a) Prove that the identity

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

for $x, y \in \mathbb{R}$,

is true, provided that $\tan^{-1} x + \tan^{-1} y$ lies in $\left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right)$.

(b) Hence evaluate

$$\tan^{-1}\left(\tfrac{1}{2}\right) + \tan^{-1}\left(\tfrac{1}{3}\right).$$

Solutions to additional exercises for Unit D4

Solution to Additional Exercise D35

(a) The domain of $f+g$ is

$$(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \cap [0,\infty) = [0, \tfrac{1}{2}\pi);$$

the rule is

$$(f+g)(x) = \tan x + \sqrt{x}.$$

(b) As in part (a), the domain of fg is $[0, \tfrac{1}{2}\pi)$; the rule is

$$(fg)(x) = \sqrt{x}\tan x.$$

(c) The domain of f/g is

$$[0, \tfrac{1}{2}\pi) - \{x : \sqrt{x} = 0\} = (0, \tfrac{1}{2}\pi);$$

the rule is

$$(f/g)(x) = \frac{\tan x}{\sqrt{x}}.$$

(d) The domain of $f \circ g$ is

$$\{x \in [0,\infty) : \sqrt{x} \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)\} = [0, \tfrac{1}{4}\pi^2);$$

the rule is

$$(f \circ g)(x) = \tan\sqrt{x}.$$

(e) The domain of $g \circ f$ is

$$\{x \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) : \tan x \in [0,\infty)\} = [0, \tfrac{1}{2}\pi);$$

the rule is

$$(g \circ f)(x) = \sqrt{\tan x}.$$

Solution to Additional Exercise D36

First we solve the equation

$$y = \frac{x-1}{x+2}$$

to obtain x in terms of y. We find that

$$y = \frac{x-1}{x+2} = 1 - \frac{3}{x+2} \iff x = -2 + \frac{3}{1-y}.$$

Thus f is one-to-one, so f has an inverse function with rule $f^{-1}(y) = -2 + 3/(1-y)$.

Now we find the image set of f, which is the domain of f^{-1}. For each $x \in (-2,\infty)$, we have $3/(x+2) > 0$, so $y < 1$. Hence $f((-2,\infty)) \subseteq (-\infty,1)$.

Also, for each $y \in (-\infty,1)$, we have $1-y>0$, so

$$x = -2 + \frac{3}{1-y} \in (-2,\infty).$$

Thus $f((-2,\infty)) \supseteq (-\infty,1)$, so

$$f((-2,\infty)) = (-\infty,1).$$

Hence the domain of f^{-1} is $(-\infty,1)$, so, adopting the usual practice of denoting the domain variable by x, we have

$$f^{-1}(x) = -2 + \frac{3}{1-x} \quad (x \in (-\infty,1)).$$

Solution to Additional Exercise D37

(a) If $0 < x_1 < x_2$, then $x_1^3 < x_2^3$ and $x_1^2 < x_2^2$ so that $-1/x_1^2 < -1/x_2^2$.

Hence

$$x_1^3 + 1 - \frac{1}{x_1^2} < x_2^3 + 1 - \frac{1}{x_2^2},$$

so f is strictly increasing on $(0,\infty)$.

(b) If $0 \le x_1 < x_2$, then $0 \le x_1^3 < x_2^3$, so

$$1 + x_1^3 < 1 + x_2^3.$$

Hence

$$\frac{1}{(1+x_1^3)^2} > \frac{1}{(1+x_2^3)^2},$$

so f is strictly decreasing on $[0,\infty)$.

Solution to Additional Exercise D38

(a) Let $g(x) = x^2+1$, $h(x) = \sin x$ and $k(x) = e^x$. Then g, h and k are basic continuous functions. Hence, by the Composition Rule,

$$f = k \circ h \circ g \text{ is continuous.}$$

(b) Let $g(x) = \sqrt{x}$, $h(x) = e^x$ and $k(x) = x^5$. Then g, h and k are basic continuous functions. Hence, by the Composition Rule and the Sum Rule,

$$f = h \circ g + k \text{ is continuous.}$$

Solution to Additional Exercise D39

(a) We prove that

$$f(x) = \begin{cases} x, & x < 0, \\ \dfrac{1}{x+1}, & x \ge 0, \end{cases}$$

is discontinuous at 0, using Strategy D14. We choose

$$x_n = -\frac{1}{n}, \quad n = 1, 2, \ldots.$$

Then $x_n \to 0$ and

$$f(x_n) = x_n \to 0.$$

Since $f(0) = 1/(0+1) = 1$, we have

$$f(x_n) \nrightarrow f(0),$$

which shows that f is discontinuous at 0.

(b) We prove that

$$f(x) = \begin{cases} \sin x \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous at 0, by using the Squeeze Rule.

Now

$$-1 \leq \sin(1/x) \leq 1, \quad \text{for } x \neq 0,$$

so

$$-\sin x \leq \sin x \sin(1/x) \leq \sin x, \qquad \text{for } x \in (0, \pi),$$

since $\sin x \geq 0$, for $x \in (0, \pi)$.

Also,

$$-\sin x \geq \sin x \sin(1/x) \geq \sin x, \qquad \text{for } x \in (-\pi, 0),$$

since $\sin x \leq 0$, for $x \in (-\pi, 0)$.

Hence

$$-|\sin x| \leq \sin x \sin(1/x) \leq |\sin x|, \qquad \text{for } 0 < |x| < \pi.$$

Since $f(0) = 0$, we deduce that

$$-|\sin x| \leq f(x) \leq |\sin x|, \quad \text{for } x \in (-\pi, \pi).$$

Thus, if we take $I = (-\pi, \pi)$, and

$$g(x) = -|\sin x| \quad \text{and} \quad h(x) = |\sin x|,$$

then condition 1 of the Squeeze Rule is satisfied.

Next, $f(0) = g(0) = h(0) = 0$, so condition 2 of the Squeeze Rule is satisfied.

Finally, the functions g and h are continuous at 0, by the Composition Rule and Multiple Rule, so condition 3 of the Squeeze Rule is satisfied.

Hence f is continuous at 0, by the Squeeze Rule.

(c) We prove that

$$f(x) = \begin{cases} (1/x)\cos(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is discontinuous at 0, using Strategy D14. We choose

$$x_n = \frac{1}{2n\pi}, \quad n = 1, 2, \ldots.$$

Then $x_n \to 0$, and

$$f(x_n) = \frac{1}{x_n}\cos\left(\frac{1}{x_n}\right) = 2n\pi \to \infty.$$

Since $f(0) = 0$, we have

$$f(x_n) \nrightarrow f(0),$$

which shows that f is discontinuous at 0.

(d) We use the Glue Rule. Let $I = \mathbb{R}$ and define the functions g and h by

$$g(x) = 0 \quad \text{and} \quad h(x) = \sqrt{x}.$$

Then f is defined on I and $0 \in I$. Also,

$$f(x) = g(x), \quad \text{for } x \in (-\infty, 0),$$
$$f(x) = h(x), \quad \text{for } x \in (0, \infty),$$

so condition 1 of the Glue Rule holds with $a = 0$.

Moreover, $f(0) = g(0) = h(0) = 0$, so condition 2 holds.

Finally, g and h are basic continuous functions so they are continuous at 0, and so condition 3 holds.

Hence f is continuous at 0, by the Glue Rule.

Solution to Additional Exercise D40

We use the Glue Rule. Take $I = (-\frac{1}{2}\pi, \infty)$ and define the functions g and h by

$$g(x) = \sin x \quad \text{and} \quad h(x) = 1.$$

Then f is defined on I and $\frac{1}{2}\pi \in I$. Also,

$$f(x) = g(x), \quad \text{for } x \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi),$$
$$f(x) = h(x), \quad \text{for } x \in (\tfrac{1}{2}\pi, \infty),$$

so condition 1 of the Glue Rule holds with $a = \frac{1}{2}\pi$.

Moreover,

$$g(\tfrac{1}{2}\pi) = f(\tfrac{1}{2}\pi) = h(\tfrac{1}{2}\pi) = 1,$$

so condition 2 holds.

Finally, g and h are basic continuous functions so they are continuous at $\frac{1}{2}\pi$, and so condition 3 holds.

Hence f is continuous at $\frac{1}{2}\pi$, by the Glue Rule.

A similar argument can be given for $a = -\frac{1}{2}\pi$, with $I = (-\infty, \frac{1}{2}\pi)$, $g(x) = -1$ and $h(x) = \sin x$.

(Alternatively, note that f is an odd function:

$$f(x) = -f(-x), \quad \text{for } x \in \mathbb{R},$$

so the continuity of f at $-\frac{1}{2}\pi$ follows from its continuity at $\frac{1}{2}\pi$, by the Composition Rule and the Multiple Rule.)

Solution to Additional Exercise D41

By calculation,

$$f(\tfrac{2}{3}\pi) = \tfrac{2}{3}\pi - \sin(\tfrac{2}{3}\pi) - \tfrac{2}{3}\pi$$
$$= -\sin(\tfrac{1}{3}\pi) = -\sqrt{3}/2 < 0$$

and

$$f(\tfrac{5}{6}\pi) = \tfrac{5}{6}\pi - \sin(\tfrac{5}{6}\pi) - \tfrac{2}{3}\pi$$
$$= \tfrac{1}{6}\pi - \sin(\tfrac{1}{6}\pi) = \tfrac{1}{6}\pi - \tfrac{1}{2} > 0,$$

because $\pi > 3$.

Also, f is continuous by the Combination Rules (since it is a sum of basic continuous functions).

Hence, by the Intermediate Value Theorem, f has a zero in $(\frac{2}{3}\pi, \frac{5}{6}\pi)$.

Solution to Additional Exercise D42

If $f(0) = 0$ or $f(1) = 1$, then we can take $c = 0$ or $c = 1$, respectively.

Otherwise, we have $f(0) > 0$ and $f(1) < 1$, since $0 \le f(x) \le 1$, for $0 \le x \le 1$.

We consider the function

$$g(x) = f(x) - x^3 \quad (x \in [0, 1]).$$

Now g is continuous on $[0, 1]$, by the Combination Rules. Moreover

$$g(0) = f(0) - 0 > 0$$

and

$$g(1) = f(1) - 1 < 0.$$

Thus, by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that

$$g(c) = 0, \quad \text{so} \quad f(c) = c^3.$$

Solution to Additional Exercise D43

(a) When

$$p(x) = x^4 - 4x^3 + 3x^2 + 2x - 1,$$

we have

$$M = 1 + \max\{|-4|, |3|, |2|, |-1|\} = 5,$$

so all the zeros of p lie in $(-5, 5)$, by Theorem D54.

Calculating $p(n)$ for integers n in $[-5, 5]$, we obtain

n	-2	-1	0	1	2	3
$p(n)$	55	5	-1	1	-1	5

Thus p changes sign on each of the intervals

$$[-1, 0], \quad [0, 1], \quad [1, 2], \quad [2, 3].$$

Since p is continuous, we deduce by the Intermediate Value Theorem that p has a zero in each of the intervals

$$(-1, 0), \quad (0, 1), \quad (1, 2), \quad (2, 3).$$

So p has at least four zeros. Since p is a polynomial of degree four, it has at most four zeros. Thus p has exactly four zeros.

(b) We have

$$p(x) = 3x^3 - 8x^2 + x + 3$$
$$= 3\left(x^3 - \tfrac{8}{3}x^2 + \tfrac{1}{3}x + 1\right).$$

For $q(x) = x^3 - \frac{8}{3}x^2 + \frac{1}{3}x + 1$ we have

$$M = 1 + \max\{|-\tfrac{8}{3}|, |\tfrac{1}{3}|, |1|\} = \tfrac{11}{3},$$

so all the zeros of q, and hence of p, lie in $(-\frac{11}{3}, \frac{11}{3})$, by Theorem D54.

Calculating $p(n)$ for integers n in $[-4, 4]$, we obtain

n	-2	-1	0	1	2	3
$p(n)$	-55	-9	3	-1	-3	15

Thus p changes sign on each of the intervals

$$[-1, 0], \quad [0, 1], \quad [2, 3].$$

Since p is continuous, we deduce by the Intermediate Value Theorem that p has a zero in each of the intervals

$$(-1, 0), \quad (0, 1), \quad (2, 3).$$

So p has at least three zeros. Since p is a polynomial of degree three, it has at most three zeros. Thus p has exactly three zeros.

Solution to Additional Exercise D44

(a) We use Strategy D15.

1. We showed that f is strictly increasing on $(0, \infty)$ in Additional Exercise D37(a).

2. The function

$$f(x) = x^3 + 1 - \frac{1}{x^2} = \frac{x^5 + x^2 - 1}{x^2} \quad (x \in (0, \infty))$$

is the restriction to $(0,\infty)$ of a rational function which is continuous on $\mathbb{R}-\{0\}$. Hence f is continuous.

3. Choose the increasing sequence (n), which tends to ∞, the right endpoint of $(0,\infty)$. Then

$$f(n)=n^3+1-\frac{1}{n^2}\to\infty \quad\text{as } n\to\infty,$$

by the Reciprocal Rule. Thus the right endpoint of $J=f((0,\infty))$ is ∞.

Then choose the decreasing sequence $(1/n)$, which tends to 0, the left endpoint of $(0,\infty)$. Then

$$f(1/n)=1/n^3+1-n^2\to-\infty \quad\text{as } n\to\infty,$$

by the Reciprocal Rule. Thus the left endpoint of J is $-\infty$.

Hence $J=(-\infty,\infty)=\mathbb{R}$, so f has a continuous inverse function

$$f^{-1}:\mathbb{R}\longrightarrow(0,\infty),$$

by the Inverse Function Rule.

(b) We use Strategy D15.

1. We showed that f is strictly decreasing on $[0,\infty)$ in Additional Exercise D37(b).
2. The function
$$f(x)=\frac{1}{(1+x^3)^2}\quad(x\in[0,\infty))$$
is the restriction to $[0,\infty)$ of a rational function which is continuous on $\mathbb{R}-\{1\}$. Hence f is continuous.
3. We have $f(0)=1$, so the corresponding endpoint of $J=f([0,\infty))$ is 1, and $1\in J$.
Now choose the increasing sequence (n), which tends to ∞, the right endpoint of $[0,\infty)$. Then
$$f(n)=\frac{1}{(1+n^3)^2}\to 0\quad\text{as } n\to\infty.$$
Thus the corresponding endpoint of J is 0, and $0\notin J$.

Hence $J=(0,1]$, so f has a continuous inverse function

$$f^{-1}:(0,1]\longrightarrow[0,\infty),$$

by the Inverse Function Rule.

Solution to Additional Exercise D45

(a) True: by definition,

$$\sin(\sin^{-1}x)=x,\quad\text{for } x\in[-1,1].$$

(b) False: for example,

$$\sin^{-1}(\sin 2\pi)=\sin^{-1}0=0.$$

Solution to Additional Exercise D46

(a) Let $a=\tan^{-1}x$ and $b=\tan^{-1}y$. Then $x=\tan a$ and $y=\tan b$, so

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}=\frac{x+y}{1-xy}.$$

Hence

$$a+b=\tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

provided that $a+b$ lies in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$, which is the image of $\tan^{-1}$. Thus

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

provided that $\tan^{-1}x+\tan^{-1}y$ lies in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$.

(b) Since $\tan^{-1}$ is strictly increasing, we have

$$0<\tan^{-1}(\tfrac{1}{2})+\tan^{-1}(\tfrac{1}{3})<2\tan^{-1}(1)=\tfrac{1}{2}\pi.$$

Hence, we can apply the formula in part (a):

$$\begin{aligned}\tan^{-1}(\tfrac{1}{2})+\tan^{-1}(\tfrac{1}{3})&=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)\\&=\tan^{-1}(1)=\tfrac{1}{4}\pi.\end{aligned}$$